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Scale-Invariant Branch Distribution from a Soluble Stochastic Model

M. B. Hastings[1]

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We consider a general model of branch competition that automatically leads to a critical branching configuration. This model is inspired by the $4-\eta$ expansion of the dielectric breakdown model, but the mechanism of arriving at the critical point may be of relevance to other branching systems as well, such as fractures. The exact solution of this model clarifies the direct renormalization procedure used for the dielectric breakdown model, and demonstrates nonperturbatively the existence of additional irrelevant operators with complex scaling dimensions leading to discrete scale invariance. The anomalous exponents are shown to depend upon the details of branch interaction; we contrast with the branched growth model in which these exponents are universal to lowest order in $1-\nu$, and show that the branched growth model includes an inherent branch interaction different from that found in the dielectric breakdown model. We consider stationary and non-stationary regimes, corresponding to different growth geometries in the dielectric-breakdown model.

KEY WORDS: Branch; fractal; growth; dielectric breakdown model.

Diffusion-limited aggregation[(1)] produces complicated fractal structures by branch competition. In this model, in radial geometry, a single seed particle is placed in a two-dimensional plane. A random walker is released from infinity, and allowed to walk until it hits the seed, at which point it sticks. Further walkers are released sequentially, with each walker only released after the previous walker has joined the growing cluster. The result of this process is a growing, branching cluster, that models a wide range of physical processes, including viscous fingering,[(2)] electrodeposition,[(3)] and dendritic growth.[(4)]

[1] CNLS, MS B258, Los Alamos National Laboratory, Los Alamos, New Mexico 87545; e-mail: hastings@cnls.lanl.gov

An interesting extension of diffusion-limited aggregation is the dielectric breakdown model.[5] In this model, a Laplacian field, ϕ, is defined surrounding the growing cluster, and the probability of adding a particle to any point on the cluster is taken to be the normal derivative of ϕ, raised to the power η. It may be shown using Green's functions that for $\eta = 1$, the process is equivalent to the diffusion-limited aggregation process. For $\eta > 1$, the probability of growth at the tips of the cluster is enhanced compared to that in diffusion-limited aggregation, and the fractal dimension decreased below the diffusion-limited aggregation value of ≈ 1.7.

Recently, a controlled renormalization group was developed within a $4-\eta$ expansion for the dielectric breakdown model.[6] This expansion is based on considering an aggregate as a collection of strictly one-dimensional branches; at $\eta = 4$, the aggregate consists of a single branch. It was shown that the probability of a single branch pair surviving for time t is proportional to $1/t^{\nu}$ (to use the terminology of the branched growth model[7]), with $1-\nu = (4-\eta)/2$. As a result, for $\eta > 4$, the probability of a branch pair surviving for time t decays faster than $1/t$, and large branch pairs are not produced. For $\eta < 4$, large branch pairs are produced. This was used to develop a direct renormalization group in an expansion in $1-\nu = (4-\eta)/2$, quantifying the mechanism of branch competition as a means of generating scale invariant clusters. It was found that, without fine tuning, the system arrives at a critical point characterized by a scale-invariant tip-splitting rate so that the renormalized probability of a branch surviving for a time t becomes proportional to $1/t$ and the rate of branch production and branch death balance at all scales. However, branching structures are common in other physical systems, and a similar $1/t$ survival probability has been found in fracture systems.[8]

Thus, we will consider a more general model of branch competition, inspired by the $4-\eta$ renormalization group, arguing that $1-\nu$ is a general expansion parameter for a wide class of branching processes. As this model is based solely on the topology of the branching process, rather than any specific geometrical properties, it may be relevant to a much wider range of branching processes than simply fractal growth. This model will be exactly solvable, clarifying the direct renormalization procedure employed for the dielectric breakdown model. We will find that the exponents depend on both ν and on the details of branch interaction, becoming trivial as the $\nu \to 1$. The exact solution of the general model reveals the presence of additional, irrelevant operators with complex scaling dimension, indicating the presence of discrete scale invariance, which has been argued to exist in diffusion-limited aggregation.[10] These operators are beyond the reach of perturbation theory.

We also compare to another model of branch competition, the branched growth model.[7] Recently,[9] it was shown that within the branched growth model the fractal dimension depends only on ν for small $1-\nu$; here, this is shown not to be true for general models of branch interaction, though the fact that the numerical values of the exponents within the $4-\eta$[6] and branched growth[9] expansions are similar is an indication that the branched growth model is a useful approximation. However, we show that the physics of the branched growth model involves certain assumptions which do not hold for the dielectric breakdown model, so that the branched growth model result is not exact for the dielectric breakdown model.

In addition to the radial geometry discussed above, diffusion-limited aggregation has also been considered in a cylindrical geometry. In this case, growth occurs on a cylinder, with the seed taken to be a straight line at the bottom of the cylinder, and particles released from above. There have been persistent questions about possible differences in fractal dimension between these two regimes. While both growth processes are non-equilibrium, in the cylindrical case, the cluster reaches a statistically stationary state, up to a trivial vertical translation, while in the radial geometry the cluster grows indefinitely.

We thus consider this model within two separate regimes. One regime will be analogous to the steady state regime in the cylindrical geometry in the dielectric breakdown model, while the other will be a non-stationary state, which corresponds to an initial condition of a single infinitely long vertical branch in cylindrical geometry, over time scales much shorter than the time required for the aggregate to reach the scale of the cylinder. We will find that the exponents characterizing local fractal dimension and mass-radius scaling are the same. This indicates that the difference between these dimensions observed in radial geometry,[11] as well as the non-trivial affine exponents observed in the early stages of growth in cylindrical geometry[12] from initial conditions consisting of a horizontal line, are due not just to the non-stationary nature of the growth but also to the different geometry and initial conditions. Another motivation to consider different geometries is to compare the branched growth model computation, performed in a non-stationary regime, to the $4-\eta$ RG computation, performed in a stationary regime.

A Model of Branch Competition. The dynamical state of the system at a given time t will be defined by a binary tree, with a set of times t_i, one for each branch point in the tree, defining the time at which that branch pair was produced. The time t is taken to be a continuous variable. For purposes of simulation later, it will be necessary to introduce an infinitesimal time step dt. There are two dynamical processes, tip-splitting which leads to the production of additional branches and branch competition.

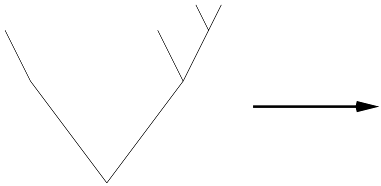

Fig. 1. Creation of a new branch point.

The first process is accounted for by assigning a rate g at which each tip (or leaf of the tree) splits, changing the topology of the tree by adding an additional branch point at that tip with time $t_i = t$. The probability of this process occurring on a given branch in a time step $\mathrm{d}t$ is $g\,\mathrm{d}t$. See Fig. 1. At the moment of branch creation, one of the two daughter branches is randomly designated as the weaker branch, and the other as the stronger (this distinction will determine which branch survives in the competition process below).

The second process is accounted for by assigning a probability per unit time of a branch point i being removed from the tree due to competition of branches. When the branch point is removed, the weaker branch is removed from the tree. See Fig. 2. We pick the probability of removing branch point i in a time step $\mathrm{d}t$ to be

$$\nu\theta(t - t_i - t_{\min}) \frac{1}{t - t_i} \mathrm{d}t \tag{1}$$

plus additional corrections due to branch interaction. Without these additional corrections, the above equation gives a probability of a weaker branch surviving (not being removed) for a time t which is proportional to $(t/t_{\min})^{-\nu}$. The θ-function in Eq. (1) plays the role of a short-time cutoff in the theory, setting the the shortest time $t_{\min}$ for which a branch may survive. Throughout we will pick

$$t_{\min} = 1 \tag{2}$$

Other choices for $t_{\min}$ do not affect the universal results.

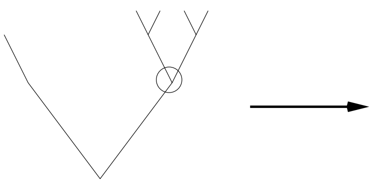

Fig. 2. Removal of the branch at the circled branch point.

To include the effects of branch interaction, we pick a phenomenological model for the interaction. We pick a scale factor $x < 1$, and declare that if on the weaker branch there exists another branch point with branching time exactly equal to $t - x(t - t_i)$ then the weaker branch will be removed. In this case, the lower branch point will have lived for a time shorter by a factor x than the branch point i.

This model has the essential features of the $4 - \eta$ renormalization group. One difference is that within that procedure the details of branch competition had to be determined numerically by integrating the trajectories of several competing branches, where here they may be determined phenomenologically by the scale factor x. Another difference has to do with the time scale. Within the dielectric breakdown model, each tip has a given growth measure and the tip-splitting rate and probability of removing branch points are proportional to the growth measure at the tip normalized by the total growth measure for the cluster. Here, we instead take all tips to have the same growth measure (so that all tips have the same probability of splitting) and we do not normalize the tip-splitting rate. The lack of normalization simply changes the overall time scale; the choice of the same growth measure for all tips is simply taken to make the model more tractable analytically and does not alter the essential physics.

In addition, in the $4 - \eta$ renormalization group, branch points may also be removed as a result of interactions with parent branches of similar scale, rather than just as a result of daughter branches. We will consider this possibility later.

Survival Probability. Let a branch pair be created at time $t = 0$. Define $s(t)$ to be the probability that the branch pair survives until time t, assuming that the branch pair is not destroyed by removal of one of its parents. We find

$$\partial_t s(t) = -\frac{v\theta(t-1)\, s(t)}{t} - (1-x)\, gs(t)\, s(xt) \tag{3}$$

Here, the first term is the change in survival probability due to the removal process in Eq. (1). The second is the change in survival probability due to branch competition, as $(1-x)\, gs(xt)\, \mathrm{d}t$ is the probability of creating another branch point at time $(1-x)\, t$ which survives until time t, causing the removal of its parent. The growth rules have been chosen so that Eq. (3) is exact; the process of creating and then removing a branch point does not affect the distribution of branches on the remaining stronger branch of that branch point.

The quantity $tgs(t)$ plays an important role. As g is the rate at which branches are created, and $s(t)$ is the survival probability of a branch, this is the probability, in time t, of creating a branch that survives for time t. This is the quantity that we will find reaches a fixed point, indicating a scaling invariant cluster. We define $tgs(t) \equiv \tilde{g}(t)$, so that $\tilde{g}(t)$ may be viewed as defining a tip-splitting rate at a time scale t.

Searching for a fixed point of Eq. (3), suppose $\tilde{g}(t) = A$. We find $A = x(1-v)/(1-x)$. The constant A sets the the dimensionless (scale invariant) tip-splitting rate.[6] As $v \to 1$, $A \to 0$. The probability of a branch being removed at exactly time t is $-\partial_t s(t) \propto t^{-2}$.

This result can also be obtained by a direct renormalization procedure[6] in which one expands the survival probability in powers of $1-v$ and g. Solving Eq. (3) to zeroth order in g and first order in $(1-v)$, $\tilde{g}(t) = A + A(1-v)\log t + \mathcal{O}(1-v)^2$. Then, solving Eq. (3) to first order in g we find $\tilde{g}(t) = A + A(1-v)\log t - A^2(1-x)\log t/(xt)$, so that a fixed point of $\tilde{g}(t)$ is reached only for the given value of A at which the logarithms on the right-hand side cancel. For the present model, the solution can be obtained exactly without the direct renormalization; we mention the direct renormalization, however, because for the dielectric breakdown model the direct renormalization procedure is required.

To investigate the approach to the fixed point, suppose instead $\tilde{g}(t) = A(1+f(t))$. Linearizing Eq. (3) about $f=0$, we find

$$\frac{\partial f(t)}{\partial \log t} = -(1-v)\, f(xt) \tag{4}$$

Equation (4) is translationally invariant in $\tau = \log t$, and has solutions $f = e^{k\tau}$, where the eigenvalue k is the scaling dimension. We find

$$k = -(1-v)\, x^k \tag{5}$$

When $v \approx 1$, the eigenvalue with largest real part is $k = -(1-v) + \mathcal{O}(1-v)^2$. For $v \approx 1$, this eigenvalue is small. In this case, we can consider the approach to scaling beyond linearization about the fixed point. We can make an approximation that $f(xt) = f(t)$ and thus approximate the non-linear problem by $\partial_t s(t) = -vs(t)/t - (1-x)\, gs^2(t)/x$, which can be solved exactly for $s(t), t > 1$ as $s(t) = (1-v)/((1-x)\, gt/x + ct^v)$, where c is an arbitrary constant.

For $1-v$ small, Eq. (5) has two solutions for real k, as well as an infinity of solutions with complex k. All these k have negative real part and describe irrelevant perturbations. For $x^{1-v} = e^{-1/e}$ the two real solutions merge, and for $x^{1-v} < e^{-1/e}$ all solutions have complex k. The presence of complex eigenvalues indicates that $s(t)$ has an oscillatory behavior and that there is a discrete scale invariance in the corrections to scaling. The scale of this discrete scale invariance is in general *not* equal to x, so that it is not simply an artifact of the particularly simple form of branch interaction chosen, but rather a result of the fact that branches separated by a finite range of scales interact. As $x^{1-\nu}$ is decreased, one finds a k with vanishing real part when $k = i\pi/(2 \log x)$, so $x^{1-\nu} = e^{-\pi/2}$. For $x^{1-\nu} < e^{-\pi/2}$, there are complex eigenvalues with positive real part which describe *relevant* perturbations so that the fixed point with $\tilde{g} = A$ becomes unstable. In this case, we have found by numerically solving Eq. (3) that $f(t)$ is driven to a new fixed point with non-decaying log-periodic oscillations, with a scale of oscillations y which is not in general equal to x. This describes a spontaneous breaking of continuous scale invariance to discrete scale invariance, $t \to yt$. Physically, this means that branches tend to be removed at certain characteristic scales, $t_0, yt_0, y^2t_0, \ldots$. The model considered here is useful for analyzing these effects, which are beyond perturbation theory in $1-\nu$.

Finally consider correlations: the probability, given that there is a branch point at time t_1 remaining in the tree until time t_1', that there is another branch point at time t_2 remaining in the tree until time t_2'. To study these, generalize the survival probability to a function $s(t, t')$, the probability that a branch point created at time t survives until time $t+t'$. We obtain the equation

$$\partial_{t'} s(t, t') = -\frac{\nu\theta(t'-1)\, s(t, t')}{t'} - (1-x)\, gs(t, t')\, s((1-x)\, t + xt', xt') \tag{6}$$

Defining $gs(t, t') = \frac{A}{t}(1 + f(t, t'))$ and linearizing we find

$$\partial_{t'} f(t, t') = -\frac{(1-\nu)}{t'} f(t + (1-x)\, t', xt') \tag{7}$$

This equation is translationally invariant in t and so we look for solutions $f(t, t') = e^{ilt} h(t')$. We find $\partial_{t'} h(t') = -\frac{(1-\nu)}{t'} h(xt')\, e^{ilxt'}$. For $l = 0$, this is the same as Eq. (5). Generally, for $t' \ll l^{-1}$, we find the same discrete scale invariant solution as for Eq. (5) as above. For $t' \gg l^{-1}$, the perturbations $f(t, t')$ decay more rapidly due to the oscillations of the exponential. Thus, the two time scales t, t' are related, and we find discrete scale invariance in both.

Simulations. It is possible to simulate the above model, and see the topology produced. A small time step dt is chosen, and the dynamics described above is simulated. We chose to plot the trees as follows. At each time step, we plot only the leaves of the tree, with the vertical position of the leaves corresponding to the elapsed time since the simulation was started. The horizontal position of the leaves is arbitrary: the model as defined above makes no reference to spatial geometry, only to topology. Thus, we introduce a rule for the horizontal position of the leaves, with the rule chosen to let the pictures accurately represent the tree. The tree is started with a single leaf at horizontal position 0. If a tip-splitting event occurs on a leaf, the daughters get positions infinitesimally displaced from the parent's position, randomly choosing which daughter is placed to the left and which to the right. Further, at every time step, each leaf is moved by a distance δ. We take δ to be proportional to the number of leafs to the left of the given leaf, minus the number of leaves to the right of the given leaf. Effectively, this introduces a repulsive force between leaves, forcing the leaves away from each other, and thus making it possible to view the tree without any problem of branches overlapping each other.

In Fig. 3 we show a simulation with $v = 0.4$, $x = 0.7$, while in Fig. 4 we show $v = 0.7$, $x = 0.7$. Other parameter choices can lead to a range of different branch densities. Figs. 3 and 4, which show the evolution of the leafs of the tree as a function of time, should be contrasted with Figs. 1 and 2

Fig. 3. Simulation with $v = 0.4$, $x = 0.7$.

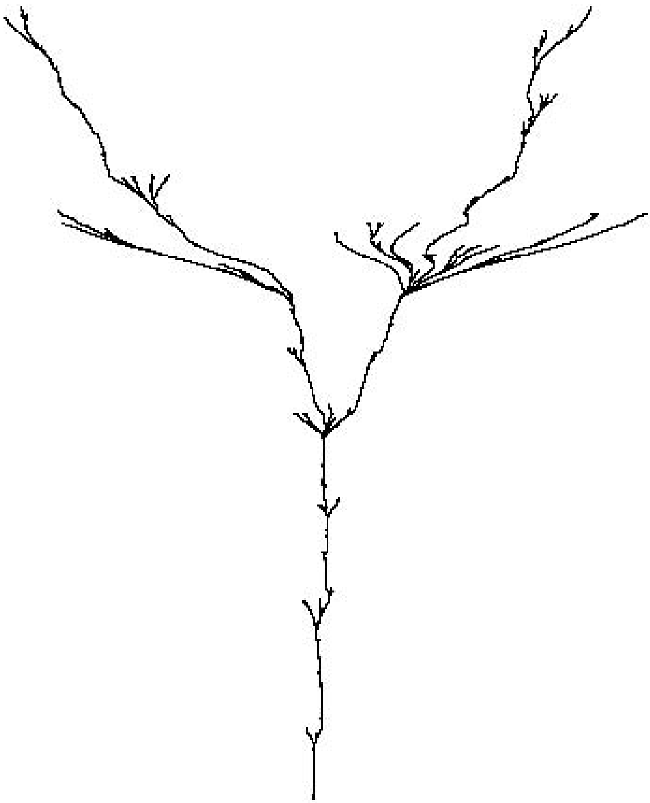

Fig. 4. Simulation with $v = 0.7$, $x = 0.7$.

which show the topology of the tree at an instant. However, as the topology of the tree at any given instant is the result of all the splitting events at preceding times, from graphs such as Figs. 3 and 4 one can determine the full topology of the tree as a function of time.

Different Geometries. We now consider the local "fractal dimension" of the cluster. Consider a subtree of the full tree consisting of a given branch below some branch point at time t_i, as well as all its subbranches due to branching events at later times. The length of a subtree is defined to be $t - t_i$. This corresponds to the vertical distance as plotted using the above procedure. The total elapsed time t sets a length scale for the cluster. The mass of a subtree is defined to be the integral, over all time, of the number of leaves in that subtree at the given time. In the case of simulations above, the integral gets replaced by a discrete sum.

Define $m(t)$ to be the average mass of a subtree at a time t after its creation, given that the subtree has not yet been removed from the tree (and assuming the subtree lie on a weaker branch). If instead the branch is removed at time t, from Eq. (3), the probability that the removal is due to processes described by Eq. (1) is v, while the probability that the removal is due to branch interaction is $1-v$. As a result, the average mass of a subtree which is removed *at* time t is $m(t)+(1-v)\,m(xt)$, where the second term represents the additional mass of the side branch in the case that removal is due to a side branch. In general, the mass $m(t)$ is equal to t, plus the mass of side branches. Side branches which appear at a time t' with $0<t'<(1-x)\,t$ can only live for a limited time, $t''<xt'/(1-x)$, or else branch competition would lead to death of the subtree. Side branches that appear at time $t'>(1-x)\,t$ can live for any time. This gives the recursion relation

$$\begin{aligned} m(t) = t + g \int_0^{(1-x)\,t} \mathrm{d}t' \int_0^{xt'/(1-x)} \mathrm{d}t'' [m(t'') + (1-v)\,m(xt'')](-\partial_t''(s(t'')) \\ + g \int_{(1-x)\,t}^{t} \mathrm{d}t' \int_0^{t-t'} \mathrm{d}t'' [m(t'') + (1-v)\,m(xt'')](-\partial_t''(s(t'')) \\ + g \int_{(1-x)\,t}^{t} \mathrm{d}t'\, m(t')\, s(t') \end{aligned} \tag{8}$$

The first two integrals are over the mass of side branches which live for time t'', after being created at time t'. The last integral is over the mass of sidebranches which remain alive at time t.

Assuming a power law $m(t)=t^D$, Eq. (8) gives $D(D-1)=A[x^{D-1}(1+(1-v)\,x^D)+x^D(D-1)]$. For $v\approx 1$, this reduces to

$$D = 1 + (1-v)\,\frac{x}{1-x} + \mathcal{O}(1-v)^2 \tag{9}$$

Consider two different geometries. If we start the tree with a single branch and follow the dynamics above, this is analogous to starting the dielectric breakdown model with a single branch as a seed configuration and letting the cluster grow. In this case, Eq. (9) provides a scaling of the mass with the time. In another geometry, analogous to the cylindrical geometry, we modify the dynamics to always remove a weaker branch if $t-t_i>T$, for some T setting a scale. In this case, after an initial non-stationary regime lasting for a time of order T, the mass of the cluster increases linearly with t, with a rate proportional to T^{D-1}: the mass of the largest branches, times the probability of producing such a branch. Thus, within this model the mass-radius scaling and local fractal dimension are the same up to a trivial difference of unity. We can generalize the model by including a possibility of removing a branch due to the presence of a parent branch of comparable size. Then, the probability of the earliest branch point i surviving till time t will scale as $(t-t_i)^{-a}\, t_i^{-b}$ where $a+b=1$. One will again find that the average time required to produce a branch point surviving for time t is of order t and the mass-radius scaling and local fractal dimension will again be equivalent.

We have checked this result for the dimension using the simulations. There are significant fluctuations about the average in individual runs, but after averaging over many runs, the dimension is recovered accurately.

Comparison to Branched Growth Model It has been shown[9] that there exists a $1-\nu$ expansion for the branched growth model similar to the $4-\eta$ expansion for the dielectric breakdown model. One elegant feature of this expansion is that the lowest order fractal dimension is obtained without considering interaction of branches, but simply from the bare constant ν. This seems surprising, as we have found within the model above, and within the $4-\eta$ expansion for the dielectric breakdown model, that unless we include branch interaction, the tip-splitting rate grows at large scales for $\nu < 1$, and a scale invariant fixed point is not reached.

The resolution of this is that the branched growth model is defined in a way which inherently includes effects of branch competition. In the branched growth model, a branch is assumed to have a probability $1/m^{\nu}$ of surviving until it reaches a *mass* m, while within the model above the probability is defined in terms of the probability to survive for a *time* t. Now, given that a scale invariant fixed point can only be reached if a branch has probability $1/t$ of surviving for time t, then we must have the relation that $1/m^{\nu} = 1/t$, so that $m = t^{1/\nu}$, and the fractal dimension is $1/\nu$.

More formally, $m(t) = t + At \log t$ to order $(1-\nu)^0, A^1$. We do not have an exact Eq. (8) for the branched growth model, but this equation is still correct to lowest order. One may still define a survival probability $s(t)$ for a branch within the branched growth model, and $\partial_t s(t) = -\nu \partial_t m(t)\, s(t)/m(t)$ so $s(t) = 1/t + (1-\nu) \log t/t - A \log t/t$ to order $(1-\nu)^1, A^1$, and we find $A = 1-\nu = D-1$.

This particular form of branch competition in the branched growth model differs from that found in the dielectric breakdown model, as found by numerically following the evolution of three branches. In some circumstances production of a daughter branch can actually reduce the competition of the parent branch with its sister;[6] it is only when summing over all configurations that the increase in branch competition is obtained. Further, the competition of branches which is inherent in the branched growth model involves only an increased competition of a branch pair due to daughter branches. However, within the $4-\eta$ expansion, it was necessary to consider the interaction of branches with daughter and parent branches (which is in fact the strongest interaction numerically) to obtain the correct result. Thus, both the present toy model and the branched growth model are approximations to the physics near $\eta = 4$, although the branched growth model dynamics serves also as a good approximation for $\eta = 1$.[7]

Conclusion. We have examined a simple, solvable model for branching, finding a fixed point with a scale invariant tip splitting rate. The direct RG for the model is exact at lowest order. The dielectric breakdown model has similar behavior and a similar perturbative RG. The exact solution enables us nonperturbatively to find additional irrelevant operators leading to a discrete scale invariance. We contrast the behavior of this model, and the dielectric breakdown model, with that of the branched growth model, for which a similar $1-\nu$ expansion is available.

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REFERENCES

1. T. A. Witten and L. M. Sander, Diffusion-limited aggregation, a kinetic critical phenomenon, *Phys. Rev. Lett.* **47**:1400 (1981).
2. J. Nittmann, G. Daccord, and H. E. Stanley, Fractal growth of viscous fingers: Quantitative characterization of a fluid instability phenomenon, *Nature* **314**:141 (1985).
3. D. Grier, E. Ben-Jacob, R. Clarke, and L. M. Sander, Morphology and microstructure in electrochemical deposition of zinc, *Phys. Rev. Lett.* **56**:1264 (1986); R. M. Brady and R. C. Ball, Fractal growth of copper electrodeposits, *Nature* **309**:225 (1984).
4. J. Kertész and T. Vicsek, Diffusion-limited aggregation and regular patterns: Fluctuations versus anisotropy, *J. Phys. A* **19**:L257 (1986).
5. L. Niemeyer, L. Pietronero, and H. J. Wiesmann, Fractal dimension of dielectric breakdown, *Phys. Rev. Lett.* **52**:1033 (1984).
6. M. B. Hastings, Growth exponents with 3.99 walkers, *Phys. Rev. E* **64**:46104 (2001); M. B. Hastings, Fractal to nonfractal phase transition in the dielectric breakdown model, *Phys. Rev. Lett.* **87**:175502 (2001).
7. T. C. Halsey and M. Leibig, Theory of branched growth, *Phys. Rev. A* **46**:7793 (1992).
8. J. P. Bouchaud *et al.*, Models of fractal cracks, *Phys. Rev. Lett.* **71**:2240 (1993).
9. T. C. Halsey, Branched growth with $\eta \approx 4$ walkers, preprint cond-mat/0105047.
10. D. Sornette *et al.*, Complex fractal dimensions describe the hierarchical structure of diffusion-limited-aggregate clusters, *Phys. Rev. Lett.* **76**:251 (1996).
11. B. B. Mandelbrot *et al.*, Deviations from self-similarity in plane DLA and the infinite drift scenario, *Europhys. Lett.* **29**:599 (1995).
12. C. Evertsz, Self-affine nature of dielectric-breakdown models clusters in a cylinder, *Phys. Rev. A* **41**:1830 (1990).